JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **76**, 213–221 (1980)

A Mean Boundary Value Problem for a Generalized Axisymmetric Potential on Doubly Connected Regions

PETER A. McCOY

Department of Mathematics, U. S. Naval Academy, Annapolis, Maryland 21402

Submitted by W. F. Ames

INTRODUCTION

In this paper an explicit expansion formula is given that recaptures a generalized axisymmetric potential (GASP), regular in a region between two concentric hyperspheres from the arithmetic means of the boundary data taken on sets of equally spaced hypercircles. Consequently, a GASP with smooth boundary data is uniquely determined by these means and a characterization of the means determines those which harmonically continue to a hypersphere. A canonical solution for the interior Dirichlet problem on doubly connected symmetric domains is then given for GASP with sufficiently smooth extension to the boundaries. The analysis follows by application of the integral operator methods of Bergman [2] and Gilbert [7, 8] to find function-theoretic generalizations of corresponding expansion formulae for analytic functions of a single complex variable made by Chui and Ching [4].

This extends a result of McCoy [11] on the solution of the interior Dirichlet problem in a hypersphere for a regular GASP with sufficiently smooth boundary data. That solution is given as a Riemann series expansion of axisymmetric harmonic polynomials whose coefficients are evaluated as the arithmetic means of the boundary values. In E^3, the Method of Ascent [8] solved the corresponding problem for a class of axisymmetric elliptic equations with entire function coefficients [3, p. 62]. The current problem does not extend to this more general class of equations on a doubly connected domain because Ascent requires that a domain be star-shaped with respect to the origin.

For constructive solutions to the interior Dirichlet problem on a hypersphere by axisymmetric interpolation harmonic polynomial approximates, refer to the papers of Marden [10] and Fryant [6]. And for relations between harmonic continuation of GASP and Chebyshev approximation to the boundary value by axisymmetric harmonic polynomials or Newtonian potentials see McCoy [12, 13].

0022-247X/80/070213–09$02.00/0

PRELIMINARIES

Let $F^\alpha = F^\alpha(x, y)$ be a regular solution of the generalized axisymmetric potential equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha + 1}{y}\frac{\partial}{\partial y}\right] F^\alpha = 0, \qquad \alpha \geqslant 0, \tag{1}$$

on a connected domain about the singular line along which the analytic Cauchy data: $F^\alpha(x, 0) = f(x)$, $(\partial F/\partial y)(x, 0) = 0$ are met. Conveniently, the domain may be viewed as its projections Ω_+ into the upper half of the complex $\mathbf{C}$-plane. [Note: $\operatorname{proj}[\Omega] = \{x + iy: (x, y) \in \Omega\}$]. The reflection $F^\alpha(x, y) = F^\alpha(x, -y)$ harmonically continues F^α as a solution of Eq. (1) on $\Omega_- = \{z \in \mathbf{C}: \bar{z} \in \Omega_+\}$ [8, 14]. The resulting harmonic function element consisting of F^α and its reflection on $\Omega \equiv \Omega_+ \cup \Omega_-$ is called a generalized axisymmetric potential. The initial domain of definition [7, 8] from which the harmonic continuations are made shall be taken as a simply connected set Ω.

On Ω, F^α is defined as the transform $F^\alpha = W_\alpha[f]$ of a unique associated analytic function f as follows:

$$\begin{gathered} F^\alpha(x, y) = \int_{\mathscr{L}_0} f(\tau)\, d\mu_\alpha(\zeta), \\ \tau = x + iy(\zeta + \zeta^{-1})/2, \qquad d\mu_\alpha(\zeta) = \lambda_\alpha(\zeta - \zeta^{-1})^{2\alpha}\, d\zeta/\zeta, \end{gathered} \tag{2}$$

where the normalization $W_\alpha[1] = 1$ is taken and the contour $\mathscr{L}_0 = \{\zeta = e^{it}: 0 \leqslant t \leqslant \pi\}$. [Note: see [12] or, for an equivalent form of W_α, see [7].] Furthermore, the inverse operator represents $f = W_\alpha^{-1}[F^\alpha]$ uniquely in terms of F^α by

$$\begin{aligned} f(z) &= \int_{\mathscr{L}_*} F^\alpha(r\xi, r(1 - \xi^2)^{1/2})\, K_\alpha(zr^{-1}, \xi)\, d\nu_\alpha(\xi), \\ K_\alpha(\eta, \xi) &= c_\alpha(1 - \eta^2)/(1 - 2\xi\eta + \eta^2)^{\alpha+3/2}, \\ d\nu_\alpha(\xi) &= (1 - \xi^2)^{2\alpha}\, d\xi, \qquad W_\alpha^{-1}[1] = 1, \end{aligned} \tag{3}$$

with contour $\mathscr{L}_* \equiv [-1, +1]$. [Note: see [7].] Each of these relations may be harmonically, and, respectively, analytically continued (together with the even extensions) from the initial domain of definition to a (larger) domain of association [7] by the Envelope Method [7, 8] as in Hadamard's multiplication of singularities theorem by continuous deformations of the contours $\mathscr{L}_0$ and $\mathscr{L}_*$ with endpoints fixed at the points $\{-1, +1\}$. The Method establishes that the domains of association of the GASP and associate are identical.

The domains of association we consider are the circle $C = C_+ \cup C_-$ and the annulus $D = D_+ \cup D_-$;

$$\begin{aligned} C_+ &= \{z \in \mathbf{C}: |z| < 1, \operatorname{Im} z \geqslant 0\}, & C_- &= \{z \in \mathbf{C}: \bar{z} \in C_+\}, \\ D_+ &= \{z \in \mathbf{C}: r_0 < |z| < 1, \operatorname{Im} z \geqslant 0\}, & D_- &= \{z \in \mathbf{C}: \bar{z} \in D_+\}. \end{aligned}$$

The domains C_+ and D_+ represent the projections of a hypersphere and region between two concentric hyperspheres. The multi-valued GASP function elements F^α that arise on D have a single branch ray on the symmetry axis. Because of the translation invariance of the GASP, it is taken along the cut $\{x + i0^+ : -\infty < x \leqslant 0\}$ in D. The principal branch of F^α is then defined as $F^\alpha(x, 0^+) = f(x + i0^+)$, corresponding to the principal branch of the associate. By the singularities theorem cited previously and the law of permanence of functional forms, the associate may be recovered on D (independently of α) by analytic continuation of the Cauchy data as

$$F^\alpha(z, 0) = f(z). \tag{4}$$

On compacta of this domain, the expansions

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{5}$$

and

$$\begin{aligned} F^\alpha(x, y) &= \sum_{n=-\infty}^{\infty} a_n V_n^\alpha(x, y), \\ V_n^\alpha(x, y) &= W_\alpha[z^n], \qquad n = 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{6}$$

represent the associate and the GASP. The function elements V_n^α may be evaluated by the Laplace-type integrals for ultraspherical polynomials [1, 2] if $n \geqslant 0$ and as a complex Maxwell expansion [7] if $n < 0$. The result is the set of multi-valued harmonic function elements

$$\begin{aligned} V_n^\alpha(x, y) &= (x^2 + y^2)^{n/2} P_n^{(\alpha,\alpha)}(x/(x^2 + y^2)^{1/2})/P_n^{(\alpha,\alpha)}(1), \qquad n \geqslant 0, \\ &= \frac{(-1)^{-n-1}}{(-n-1)!} \left(\frac{\partial}{\partial \gamma}\right)^{-n-1} [W_\alpha[(z - \gamma)^{-1}]] \,|_{\gamma=0}\,, \qquad n < 0, \end{aligned}$$

on the initial domain of definition, where the $P_n^{(\alpha,\alpha)}$ are the ultraspherical polynomials. In particular, for the axisymmetric potentials in E^3, $\alpha = 0$ and

$$\begin{aligned} V_n^0(x, y) &= (x^2 + y^2)^{n/2} P_n(x/(x^2 + y^2)^{1/2}), \qquad n \geqslant 0, \\ &= \frac{(-1)^{-n-1}}{(-n-1)!} \left(\frac{\partial}{\partial \gamma}\right)^{-n-1} [(x - \gamma)^2 + y^2]^{1/2} \,|_{\gamma=0}\,, \qquad n < 0, \end{aligned}$$

where the P_n are the Legendre polynomials. For non-integral 2α, V_{-n}^α may be evaluated by Legendre functions as in [7] or by fractional integrals [9]. This brings us to our main objective.

The Boundary Value Problem

The class $A^{1+\epsilon}(D)$ designates those regular F^{α} on D for which

$$a_n = O(1/n^{1+\epsilon}), \qquad a_{-n} = O(r_0/n^{1+\epsilon}), \qquad n > 0,$$

for some $\epsilon > 0$. Following the development for the analytic functions [4] on D, only those F^{α} whose associates are continuous on $\mathrm{cl}(D)$ are considered. This is a sufficient condition for continuity of F^{α} on $\mathrm{cl}(D)$ in view of the definition of the operator W_{α}.

The multi-valued generalized axisymmetric annual harmonic function elements that form the a basis for the solution of the mean boundary value problem are

$$Q_n^{\alpha}(x, y) = \sum_{m/n} \frac{\mu(n/m)}{r_0^{-m} - r_0^{m}} \left[\frac{V_m^{\alpha}(x, y)}{r_0^{m}} - \frac{V_{-m}^{\alpha}(x, y)}{r_0^{-m}}\right] \tag{7}$$

and

$$Q_{0,n}^{\alpha}(x, y) = \sum_{m/n} \frac{\mu(n/m)}{r_0^{m} - r_0^{-m}} [V_m^{\alpha}(x, y) - V_{-m}^{\alpha}(x, y)], \tag{8}$$

$n = 1, 2, \ldots$. The function $\mu(n)$ is the Möbius function and n/m means that n is a factor of m. The expansion coefficients for $F^{\alpha} \in A^{1+\epsilon}(D)$ are evaluated by the arithmetic means taken from the Cauchy data as

$$\sigma_n(F^{\alpha}) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{F}^{\alpha}(\cos 2\pi k/n, \sin 2\pi k/n) \tag{9}$$

and

$$\sigma_{0,n}(F^{\alpha}) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{F}^{\alpha}(r_0 \cos 2\pi k/n, r_0 \sin 2\pi k/n) \tag{10}$$

for $F^{\alpha}(re^{i\theta}, 0) = \mathbf{F}^{\alpha}(r\cos\theta, r\sin\theta)$ taken at the points $\omega_n^{k} = e^{(2\pi k/n)i}$, $0 \leqslant k \leqslant n$, $n = 1, 2, \ldots$, which are interpreted as hypercircles in view of the connection between Eq. (1) and the axisymmetric Laplace equation in E^q for $2\alpha = q - 3$, $q = 3, 4, \ldots$. The limiting averages

$$\sigma_{\infty}(F^{\alpha}) = \lim_{n\to\infty} \sigma_n(F^{\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{F}^{\alpha}(\cos s, \sin s)\, ds$$

and

$$\sigma_{0,\infty}(F^{\alpha}) = \lim_{n\to\infty} \sigma_{0,n}(F^{\alpha}) = \frac{1}{2\pi r_0} \int_0^{2\pi r_0} \mathbf{F}^{\alpha}(r_0 \cos s, r_0 \sin s)\, ds,$$

where $F^{\alpha}(e^{i\theta}, 0) = F^{\alpha}(\cos\theta, \sin\theta)$ satisfy the relation $r_0\sigma_{\infty}(F^{\alpha}) = \sigma_{0,\infty}(F^{\alpha})$ by [4]. The so-called Riemann coefficients of F^{α}

$$\rho_n(F^{\alpha}) = \sigma_n(F^{\alpha}) - \sigma_{\infty}(F^{\alpha}), \qquad \rho_{0,n}(F^{\alpha}) = \sigma_{0,n}(F^{\alpha}) - \sigma_{0,\infty}(F^{\alpha}), \tag{11}$$

$n = 1, 2, \ldots$, evaluating the remainders of the series are also required. Because of Eq. (4), these means and coefficients are in fact independent of α. The errors in approximate solutions are taken in the sup-norm $\|\cdot\|$, which is

$$\|F^\alpha\| = \sup\{|F^\alpha(x, y)| : x + iy \in \mathrm{cl}(D)\}.$$

The following representation formula stands.

THEOREM 1. *Let the principal branch of the* GASP *function element* $F^\alpha \in A^{1+\epsilon}(D)$ *for some* $\epsilon > 0$. *Then the principal branch of the harmonic function element*

$$\sum_{n=1}^{\infty} \rho_n(F^\alpha) Q_n^\alpha(x, y) + \sum_{n=1}^{\infty} \rho_{0,n}(F^\alpha) Q_{0,n}^\alpha(x, y) + \sigma_x(F^\alpha) \tag{12}$$

is a uniformly convergent expansion of F^α *on compacta of* D *for all* $\alpha \geqslant 0$. *The error in approximating* F^α *by the principal branch of the* mth *partial sum of the expansion is*

$$\left\| F^\alpha - \sum_{n=1}^{m} \rho_n(F^\alpha) Q_n - \sum_{n=1}^{m} \rho_{0,n}(F^\alpha) Q_{0,n}^\alpha - \sigma_x(F^\alpha) \right\| = O(1/m^\delta),$$

$m = 1, 2, \ldots$, *for any fixed* δ, $0 < \delta < \epsilon$.

Proof. Each $F^\alpha \in A^{1+\epsilon}(D)$ has a unique representation $F^\alpha = W_\alpha[f]$, generated by deforming the contour $\mathscr{L}_0$ as necessary so that the vector $\tau \in D$. The associate f is itself a number of the class $A^{1+\epsilon}(D)$. Therefore, a sequence of analytic functions $\{s_m\}$ is constructed as in [4] by

$$s_m(z) = \sum_{n=1}^{m} \rho_n(F^\alpha)\, q_n(z) + \sum_{n=1}^{m} \rho_{0,n}(F^\alpha)\, q_{0,n}(z) + \sigma_x(F^\alpha), \qquad m = 1, 2, \ldots,$$

where

$$q_n(z) = \sum_{m/n} \frac{\mu(n/m)}{r_0^{-m} - r_0^{m}} \left\{ \left(\frac{z}{r_0}\right)^m - \left(\frac{z}{r_0}\right)^{-m} \right\}, \tag{13}$$

$$q_{0,n}(z) = \sum_{m/n} \frac{\mu(n/m)}{r_0^{m} - r_0^{-m}} \{z^m - z^{-m}\}, \tag{14}$$

with the properties that: (i) $\{s_m\}$ is uniformly convergent on compacta of $\mathrm{cl}(D)$, (ii) the restriction of $s \equiv \lim_{m\to\infty} s_m$ to ∂D and the restriction of f to ∂D are identical. Hence, by the maximum principle $s \equiv f$ on $\mathrm{cl}(D)$.

For each n, select the branch of the harmonic function elements

$$Q_n^\alpha = W_\alpha[q_n], \qquad Q_{0,n}^\alpha = W_\alpha[q_{0,n}]$$

for which $W_\alpha[s_m]$ is the principal value of $S_m{}^\alpha = W_\alpha[s_m]$. The bound

$$\| S^\alpha - S_m{}^\alpha \| \leqslant \| f - s_m \| \left\{ \int_{\mathscr{L}} | d\mu_\alpha(\zeta) | \right\} \tag{15}$$

and properties of $\{s_m\}$ verify that $S^\alpha = \lim_{m\to\infty} S_m{}^\alpha$ is a GASP on D and continuous on cl(D). Then the maximum principal shows that $S^\alpha = F^\alpha$ on cl(D). To establish the estimate on the remainder, apply to Eq. (15) the classical result [4] which states that $\| f - s_m \| = O(1/m^\delta)$ for $\epsilon > 0$ and δ fixed, $0 < \delta < \epsilon$, $m = 1, 2, \ldots$.

A direct consequence of this representation formula is the following uniqueness characterization. It is obtained by applying the reasoning in [4] to the representaiton formula above.

Corollary 1.1. *Let the principal branch of the* GASP *function element* F^α *be in class* $A^{1+\epsilon}(D)$ *for some* $\epsilon > 0$ *and order* α. *If the arithmetic means*

$$\sigma_n(F^\alpha) = \sigma_{0,n}(F^\alpha) = 0, \qquad n = 1, 2, \ldots,$$

then for each order $\alpha \geqslant 0$, *the* GASP $F^\alpha \equiv 0$ *on* cl(D).

The identification of those GASP that have a harmonic continuation from D to its open convex hull occurs in terms of the complete set of harmonic polynomials

$$H_m{}^\alpha(x, y) = \sum_{s/m} \mu(m/s)\, V_s{}^\alpha(x, y), \qquad m = 1, 2, \ldots, \tag{16}$$

$H_0{}^\alpha(x, y) = 1$, introduced in [11].

Corollary 1.2. *Let the principal branch of the* GASP *function element* F^α *be in class* $A^{1+\epsilon}(D)$ *for some* $\epsilon > 0$ *and order* α. *Then* F^α *has a harmonic continuation to the open convex hull of* D *if, and only if,*

$$\rho_{0,n}(F^\alpha) = \sum_{m=1}^{\infty} H_m{}^\alpha(r_0{}^n, 0)\, \rho_{n,m}(F^\alpha), \qquad n = 1, 2, \ldots. \tag{17}$$

Proof. If F^α has harmonic continuation to $\mathbf{D}$, the open convex hull of D, its associate analytically continues to $\mathbf{D}$. Since the associate F is analytic independent of α, each F in the family harmonically continues to D. These are consequences of the singularities theorem. The analyticity of $\mathbf{F}^\alpha$ on D may be stated in terms of its Riemann coefficients as

$$\rho_{0,n}(F^\alpha) = \sum_{m=1}^{\infty} p_m(r_0{}^n)\, \rho_{n,m}(F^\alpha), \qquad n = 1, 2, \ldots, \tag{18}$$

with

$$p_m(z) = \sum_{s/m} \mu(m/s)\, z^s \tag{19}$$

[see 4]. However, $H_m{}^\alpha(z, 0) = p_m(z)$, $z \in \mathbf{C}$, and $H_m{}^\alpha(x, y) = W_\alpha[p_m]$ so that (18) and (19) are equivalent and (17) follows. Clearly, if (18) then f continues analytically to $\mathbf{D}$, F^α harmonically continues to $\mathbf{D}$ and (17) is satisfied.

As an alternate form of Theorem 1, let us define for each n,

$$h_n{}^+ = h_n{}^+(\cos\theta) \qquad \text{and} \qquad h_n{}^- = h_n{}^-(r_0 \cos\theta),$$

the non-decreasing Heaviside functions with unit jumps at the hypercircles along the intersection of the annulus $\mathrm{cl}(D)$ and the hyperplanes $x_{k,n} = \cos(2\pi k/n)$, $k = 1,\ldots, n-1$. By selecting the normalizations $h_n{}^+(1^+) = h_n{}^-(1^+) = 1$, formulas (12) may be rewritten as averages over hyperspheres using Stieltjes integrals in the following way:

$$\sigma_n(F^\alpha) = n \int_0^\pi F^\alpha(\cos\phi, \sin\phi)\, G_n{}^+(\cos\phi)\, d\phi,$$

$$\sigma_{0,n}(F^\alpha) = n \int_0^\pi F^\alpha(\cos\phi, \sin\phi)\, G_n{}^-(\cos\phi)\, d\phi,$$

where

$$h_n{}^+(\cos\theta) = \int_1^{\cos\theta} G_n{}^+(t)\, dt + 1$$

and

$$h_n{}^-(r_0\cos\theta) = \int_1^{r_0\cos\theta} G_n{}^-(t)\, dt + 1.$$

Now then, the measures defined as

$$dg_+{}^\alpha(x, y, \cos\theta) = \sum_{n=1}^\infty nQ_n{}^\alpha(x, y)\, G_n{}^+(\cos\theta)\, d\theta \tag{20}$$

and

$$dg_-{}^\alpha(x, y, \cos\theta) = \sum_{n=1}^\infty nQ_{0,n}^\alpha(x, y)\, G_n{}^-(\cos\theta)\, d\theta \tag{21}$$

play the role of Green's functions. Thus Theorem 1 appears in equivalent form as

THEOREM 2. *Let the principal branch of the* GASP *function element* F^α, *a member of class* $A^{1+\epsilon}(D)$ *for some* $\epsilon > 0$ *and* α, *be normalized by* $\sigma_\infty(F^\alpha) = \sigma_{0,\infty}(F^\alpha)$

$= 0$. *Then for each order* $\alpha \geq 0$, *the principal branch of the harmonic function element is represented by*

$$F^{\alpha}(x, y) = \int_0^{\pi} \mathbf{F}^{\alpha}(\cos\phi, \sin\phi)\, dg_{+}^{\alpha}(x, y, \cos\phi)$$

$$+ \int_0^{\pi} \mathbf{F}^{\alpha}(\cos\phi, \sin\phi)\, dg_{-}^{\alpha}(x, y, \cos\phi)$$

uniformly on compacta of cl(D).

To illustrate a broader application of these results, we close with an extension to a general class of doubly connected sets Ω about the origin that are symmetric $(\Omega = \bar{\Omega})$ and have smooth boundary. Specifically, those sets Ω that are smooth in the following sense: (i) let Ω be the image of D under an explicit conformal map $\phi\colon \Omega \to D$ and (ii) let ϕ and ϕ^{-1} extend continuously to cl(Ω) and cl(D) as smooth functions. Then we have the following.

THEOREM 3. *Let* Ω *be a smooth doubly connected domain and let the conformal map* $\phi\colon \Omega \to D$ *for which* $\phi \in C'(\mathrm{cl}(\Omega))$ *and* $\phi^{-1} \in C'(\mathrm{cl}(D))$. *Let the principle branch of the* GASP *function element* F^{α} *be in* $A^{1+\epsilon}(D)$ *for some* $\epsilon > 0$ *and order* α. *Then for each order* $\alpha \geq 0$, *the principal branch of the* GASP *function element regular on* Ω *with boundary values* $W_{\alpha}[F \circ \phi^{-1}]$ *is given by*

$$H^{\alpha}(x, y) = \sum_{n=1}^{\infty} \rho_n(\mathbf{H}^{\alpha})\, W_{\alpha}[q_n \circ \phi^{-1}] + \sum_{n=1}^{\infty} \rho_{0,n}(\mathbf{H}^{\alpha})\, W_{\alpha}[q_{0,n} \circ \phi^{-1}] + \sigma_{\infty}(\mathbf{H}^{\alpha}) \tag{21}$$

uniformly on compacta of cl(Ω) *for* $\mathbf{H}^{\alpha} = \mathbf{F}^{\alpha} \circ \phi^{-1}$.

Proof. Let $s_m \Rightarrow s$ on cl(D) as in Theorem 1 and define the sequence of functions $h_m = s_m \circ \phi^{-1}$ analytic on Ω and continuous on cl(Ω). The function $\mathbf{H}^{\alpha} \equiv s \circ \phi^{-1} = \lim_{m \to \infty} h_m$ is analytic on Ω and continuous of cl(Ω). Therefore, $H^{\alpha} = W_0(\mathbf{H}^{\alpha})$ is a GASP on Ω with boundary values $W_{\alpha}[s \circ \phi^{-1}]$. Since all limiting operations are uniformly convergent on compacta of cl(Ω), by replacing $Q_n{}^{\alpha} = W_{\alpha}[q_n \circ \phi^{-1}]$ and $Q_{0,n} = W_{\alpha}[q_{0,n} \circ \phi^{-1}]$ and $\rho_n(\rho_n(F \circ \phi^{-1})$ and $\rho_{0,n}(F^{\alpha} \circ \phi^{-1})$ in (12), the theorem is proved by observing that $s^{\alpha} = F^{\alpha}$ on cl(D).

We close by remarking that the range of orders $-\frac{1}{2} < \alpha < 0$ was not included in this study due to nontrivial representations of the identity for these values [see [5]].

ACKNOWLEDGMENT

This research was partially funded by a grant from the U. S. Naval Academy Research Council.

References

1. R. Askey, "Orthogonal Polynomials and Special Functions," Regional Conference Series in Applied Math., SIAM, Philadelphia, 1975.
2. R. Askey Ed., "Theory and Application of Special Functions," Academic Press, New York, 1975.
3. S. Bergman, "Integral Operators in the Theory of Linear Partial Differential E-quations," Ergebnisse der Math. Heft 23, Springer-Verlag, New York, 1961.
4. C. H. Ching and C. K. Chui, Recapturing a holomorphic function on an annulus from its mean boundary values, *Proc. Amer. Math. Soc.* **41**, No. 1 (1973), 120–126.
5. D. L. Colton, Applications of a class of singular partial differential equations to Gegenbauer series which converge to zero, *SIAM J. Math. Anal.* **1**, No. 1 (1970), 90–95.
6. A. J. Fryant, Interpolation and approximation of axisymmetric harmonic functions, *Amer. J. Math.* **100**, No. 1 (1978), 205–216.
7. R. P. Gilbert, "Function Theoretic Methods in Partial Differential Equations," Mathematics in Science and Engineering, Vol. 54, Academic Press, New York, 1969.
8. R. P. Gilbert, "Constructive Methods for Elliptic Equations," Lecture Notes in Mathematics No. 365, Springer-Verlag, New York, 1974.
9. J. L. Lavoie, T. J. Osler, and R. Trembly, Fractional derivatives and special functions, *SIAM Rev.* **18**, No. 2 (1976), 240–268.
10. M. Marden, Axisymmetric harmonic interpolation polynomials in R^n, *Trans. Amer. Math. Soc.* **196** (1974), 385-402.
11. P. A. McCoy, Mean boundary value problems for a class of elliptic equations in E^3, *Proc. Amer. Math. Soc.* **76**, 1 (1979), 123–128.
12. P. A. McCoy, Polynomial approximation and growth of generalized axisymmetric potentials, *Canad. J. Math.* **31**, No. 1 (1979), 49–59.
13. P. A. McCoy, Approximation and harmonic continuation of axially symmetric potentials in E^3, *Pacific J. Math.* **90**, No. 2 (1979), 481–491.
14. B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–91.